UCID-21698

PROBABILITY DISTRIBUTIONS FOR WEAPON SYSTEM EFFECTIVENESS

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MARCH 22, 1989

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Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under Contract W-7405-Eng-48.

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Printed in the United States of America
Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161

Price Code	Page Range
A01	Microfiche
Papercopy Prices	
A02	001-050
A03	051-100
A04	101-200
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A06	301-400
A07	401-500
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Probability Distributions for Weapon System Effectiveness

Abstract

I derive the probability density functions and cumulative distribution functions describing the single shot probability of survival, SSPS, for a given weapon system and target, as a function of the underlying random variables weapon radius, WR, and circular error probable, CEP. I derive explicit analytical formulas when WR and CEP are uniformly distributed and numerically compute results when WR is unform and CEP is distributed as χ^2. I illustrate some properties of the SSPS distributions and how these results can apply to weapon effectiveness studies.

1. Motivation

The single shot probability of survival, which I denote here by s, and its complement the single shot probability of kill, denoted here by p, are key measures of weapon system effectiveness, given by [1]

$$s(u,v) = \left[\tfrac{1}{2}\right]^{\left(\frac{u}{v}\right)^2}, \tag{1}$$

$$p(u,v) = 1 - s(u,v). \tag{2}$$

Here u is the weapon radius, which expresses how close to the target the weapon must land to destroy the target and v is the circular error probable, which measures how accurately the weapon system delivers its weapon. Reference [1] gives more rigorous and precise definitions of these quantities. (Traditonally, s,p,u, and v are denoted SSPS, SSPK, WR and CEP respectively. Here and below we abbreviate the notation where necessary to clarify the presentation.) Calculations of u combine weapon effects phenomena with target characteristics while calculations of v involve delivery system characteristics, and the results for both often have unavoidable uncertainties. Due to the uncertainties, studies must consider s for a distribution of possible values of u and v and typically choose two from a handful of common probability distributions for computation. Hence s is a function of random variables and in principle is a random variable with a distribution of its own, derivable from the distributions of u and v.

Many studies require knowledge of the distribution of s 1) to understand the relationship between means, quantiles and variances of u and v and the means, quantiles, and variances of s (and p) 2) to test hypotheses and goodness of fit of models in cases where input data with considerable uncertainty has been used, and 3) to estimate effects of changes in value of underlying parameters. Treatments of this kind of problem range from Monte Carlo simulation [2] to structured sampling from discretized distributions [3]. Here we obtain cumulative distribution functions from knowledge of the underlying distributions of WR and CEP and focus on convenient estimates of confidence levels of the SSPS. By providing ready estimates of confidence levels, these results complement other approaches to SSPS estimation.

In Section 2 below I derive the probability density function (pdf) and the cumulative distribution function (cdf) of s(u,v) defined as a function of random variables u and v for the case in which u and v are both distributed uniformly. Section 3 treats the case in which v is distributed as chi-square and u is distributed uniformly. Section 4 discusses the case in which v is distributed a chi-square and u follows a general, possibly quasi-empirical distribution.

2. Pdf and cdf for u and v uniform.

Let u and v be independent and uniformly distributed, with pdfs given by

$$f_u(u) = 1/R_u \quad , \quad 0 < u_L \le u \le u_H \quad , \quad R_u = (u_H - u_L) \tag{3}$$
$$= 0 \qquad \text{elsewhere}$$
$$f_v(v) = 1/R_v \quad , \quad 0 < v_L \le v \le v_H \quad , \quad R_v = (v_H - v_L) \tag{4}$$
$$= 0 \qquad \text{elsewhere.}$$

Here, the subscripts H and L on u and v indicate the upper and lower limits (High and Low) of those variables. The cdf of s(u,v) is given by integrating the joint pdf of u and v over the range of (u,v) such that $0 \le \left(\frac{1}{2}\right)^{\left(\frac{u}{v}\right)^2} \le s$ namely [see 4 for a discussion of similar cases]

$$F(s) = \iint\limits_{(u,v):\, \left(\frac{1}{2}\right)^{(u/v)^2} \le s} f_u(u)\, f_v(v)\, du\, dv = \int_0^{\infty} du\, f_u(u) \int\limits_{v:\, s(u,v) \le s} f_v(v)\, dv \tag{5}$$

The integral over v can be rewritten as an integral over s by using the inverse relation

$$v(u,s) = u\left(\frac{\ln(1/2)}{\ln(s)}\right)^{1/2} , \tag{6}$$

$$dv = -\frac{u}{2\ln(1/2)}\left(\frac{\ln(1/2)}{\ln(s)}\right)^{3/2}\frac{ds}{s}$$

and noting that the relation between s and v is well behaved everywhere. Then the integral for F(s) becomes

$$F(s) = -\frac{1}{2\ln(1/2)}\int_0^{\infty} du\int_0^{s} u\, f_u(u)\, f_v(v(u,s'))\left[\frac{\ln(s')}{\ln(1/2)}\right]^{-3/2}\frac{ds'}{s'} \tag{7}$$

The pdf for s is then given by

$$f_s(s) = \frac{dF(s)}{ds} = -\frac{1}{2\ln(1/2)}\left[\frac{\ln(s)}{\ln(1/2)}\right]^{-3/2}\frac{1}{s}\int_0^{\infty} u\, f_u(u)\, f_v(v(u,s))\, du \tag{8}$$

Since u is uniform this becomes

$$f_s(s) = -\frac{1}{2R_u\ln(1/2)}\frac{1}{s}\left(\frac{\ln(s)}{\ln(1/2)}\right)^{-3/2}\int_{u_L}^{u_H} u f_v(v(u,s))\, du \tag{9},$$

$$= -\frac{1}{4R_uR_v\ln(1/2)}\frac{1}{s}\left(\frac{\ln(s)}{\ln(1/2)}\right)^{-3/2} u^2\Big|_{u(\text{lower limit})}^{u(\text{upper limit})} \tag{10}$$

with the implicit definition

$$f_v(v(u,s)) = 1/R_v\,,\quad v_L \le v(u,s) \equiv u\left(\frac{\ln(1/2)}{\ln(s)}\right)^{1/2} \le v_H \tag{11}$$

$$= 0 \qquad \text{elsewhere,}$$

and the upper and lower limits are functions of s, depending on the non-null intersection of (3) and (11); the result is the following.

F(s) and f(s) are described completely by the four parameters, u_H, u_L, v_H, and v_L. In particular, they define four values of s on the interval [0,1]

$$s_1 = \left(\tfrac{1}{2}\right)^{\left(\frac{u_H}{v_L}\right)^2} \qquad s_2 = \left(\tfrac{1}{2}\right)^{\left(\frac{u_H}{v_H}\right)^2}$$

$$s_3 = \left(\tfrac{1}{2}\right)^{\left(\frac{u_L}{v_L}\right)^2} \qquad s_4 = \left(\tfrac{1}{2}\right)^{\left(\frac{u_L}{v_H}\right)^2} \tag{12}$$

There are two cases to consider, depending on which of u or v is more uncertain:

Case (a): $$\frac{v_H}{v_L} \leq \frac{u_H}{u_L} \tag{13}$$

In this case the range of the v (CEP) distribution is less than that of the u distribution, the most interesting case in practice. Here we have $s_1 \leq s_2 \leq s_3 \leq s_4$. Box 1 gives the pdf and cdf ($f_s(s)$ and F(s)) for case (a). Elementary integration of the pdf provides the cdf. The median, mean, and variance can be calculated explicitly as well. The median is found by noting that (13) requires the median, s_{50} , to be in the interval $s_2 \leq s \leq s_3$. Setting the cdf F(s), Box 1, to 1/2 yields

$$\frac{1}{2} = F(s_{50}) = \left[\frac{u_H}{v_H} - \left(\frac{\ln(s_{50})}{\ln(1/2)}\right)^{1/2}\right]\left[\frac{v_h^{\,2} - v_L^{\,2}}{2R_uR_v}\right] + F(s_2) \tag{14}$$

solving for the median gives

$$s_{50} = \left(\tfrac{1}{2}\right)^{\left(\frac{u_H+u_L}{v_H+v_L}\right)^2} = \left(\tfrac{1}{2}\right)^{\left(\frac{u_{50}}{v_{50}}\right)^2} = \left(\tfrac{1}{2}\right)^{\left(\frac{<u>}{<v>}\right)^2} \tag{15}$$

where the bracketing denotes the mean. Thus for the special case of uniformly distributed u and v, the median of s is s evaluated at the medians or the means, since the median and mean of the uniform distribution are identical. The mean is found by integration,

$$\langle s\rangle = \int s\, f_s(s)\, ds \tag{16}$$

where the pdf is given in Box 1. The result is given in Box 2. We obtain the second moment of s, $\langle s^2\rangle$, by substituting s^2 for s in (16), the result being similar in form to the expression for the mean. The result is included in Box 2. Finally, we can solve for an arbitrary q-quantile by setting F(s)=q and solving for s, being careful to assure the result is appropriate to the interval of s used. The results for q-quantiles is also in Box 2

Case (b): $\frac{v_H}{v_L} \geq \frac{u_H}{u_L}$.

This case is less interesting and very similar, hence is not discussed at length here. I give the pdf and cdf for reference in Box 3 as an appendix.

BOX 1

Case (a): $\frac{v_H}{v_L} \le \frac{u_H}{u_L}$

$$f(s) = 0 \qquad s \le s_1$$

$$f(s) = Q(s)\left[u_H^2 - v_L^2\left(\frac{\ln(s)}{\ln(1/2)}\right)\right] \qquad s_1 \le s \le s_2$$

$$f(s) = Q(s)\left[v_H^2 - v_L^2\right]\left(\frac{\ln(s)}{\ln(1/2)}\right) \qquad s_2 \le s \le s_3$$

$$f(s) = Q(s)\left[v_H^2\left(\frac{\ln(s)}{\ln(1/2)}\right) - u_L^2\right] \qquad s_3 \le s \le s_4$$

$$f(s) = 0 \qquad s_4 \le s$$

$$Q(s) \equiv -\frac{1}{4R_uR_v\ln(1/2)}\left(\frac{1}{s}\right)\left(\frac{\ln(1/2)}{\ln(s)}\right)^{3/2}$$

$$F(s) = 0 \qquad 0 \le s \le s_1$$

$$F(s) = \frac{1}{2R_uR_v}\left\{u_H^2\left(\frac{\ln(s)}{\ln(1/2)}\right)^{-1/2} - 2u_Hv_L + v_L^2\left(\frac{\ln(s)}{\ln(1/2)}\right)^{1/2}\right\}$$
$$s_1 \le s \le s_2$$

$$F(s) = \left[\frac{u_H}{v_H} - \left(\frac{\ln(s)}{\ln(1/2)}\right)^{1/2}\right]\left[\frac{v_H^2 - v_L^2}{2R_uR_v}\right] + F(s_2) \qquad s_2 \le s \le s_3$$

$$F(s) = \frac{1}{2R_uR_v}\left\{u_L\left[v_L + \frac{v_H^2}{v_L}\right] - v_H^2\left(\frac{\ln(s)}{\ln(1/2)}\right)^{1/2} - u_L^2\left(\frac{\ln(s)}{\ln(1/2)}\right)^{-1/2}\right\} + F(s_3)$$
$$s_3 \le s \le s_4$$

$$F(s) = 1 \qquad s_4 \le s \le 1$$

BOX 2

Case (a): $\dfrac{v_H}{v_L} \le \dfrac{u_H}{u_L}$

$$s_{50} = \left(\tfrac{1}{2}\right)^{\left(\frac{u_H+u_L}{v_H+v_L}\right)^2} = \left(\tfrac{1}{2}\right)^{\left(\frac{u_{50}}{v_{50}}\right)^2} = \left(\tfrac{1}{2}\right)^{\left(\frac{\langle u\rangle}{\langle v\rangle}\right)^2}$$

$$\langle s\rangle = \frac{1}{4R_uR_v}\{\Gamma(\tfrac{1}{2}, x_1)\left[v_v^2a^{-1} + 2au_H^2\right] - \Gamma(\tfrac{1}{2}, x_2)\left[v_H^2a^{-1} + 2u_H^2a\right] - \Gamma(\tfrac{1}{2}, x_3)\left[v_L^2a^{-1} + 2u_L^2a\right]$$

$$+ \Gamma(\tfrac{1}{2}, x_4)\left[v_H^2a^{-1} + 2u_L^2a\right] + 2u_H^2a\left[x_2^{-1/2}e^{-x_2} - x_1^{-1/2}e^{-x_1}\right] + 2u_L^2a\left[x_3^{-1/2}e^{-x_3} - x_4^{-1/2}e^{x_4}\right]\}$$

where $a = |\ln(1/2)|^{1/2}$, $x_n = -\ln(s_n)$, $n = 1,2,3,4$

and the incomplete gamma function is defined as

$$\Gamma(v,at) = a^v\int_t^{\infty} r^{v-1}e^{-at}dr \qquad a > 0, t > 0$$

$$s_q = \left(\tfrac{1}{2}\right)^{\left[\frac{qR_uR_v+u_Hv_L-\left(q^2R_u^2R_v^2+2qR_uR_vu_Hv_L\right)^{1/2}}{v_L^2}\right]^2} \qquad s_1 \le s_q \le s_2, \qquad q \le \frac{1}{2}\left(\frac{1-\frac{v_L}{v_H}}{1-\frac{u_L}{u_H}}\right)$$

$$s_q = \left(\tfrac{1}{2}\right)^{\left[\frac{\left[2\langle u\rangle-\left(qu_H+(1-q)u_L\right)\right]}{|v|}\right]^2} \qquad s_2 \le s_q \le s_3, \qquad q \le \left(\frac{u_Hv_L^2 - u_L^2\langle v\rangle}{R_uv_L^2}\right)$$

$$s_q = \left(\tfrac{1}{2}\right)^{\left[\frac{qR_uR_v-u_Lv_L+u_HR_v-\left[\left(qR_uR_v-u_Lv_L+u_HR_v\right)^2-u_L^2v_H^2\right]^{1/2}}{v_H^2}\right]^2} \qquad s_3 \le s_q \le 1, \left(\frac{u_Hv_L^2-u_L^2\langle v\rangle}{R_uv_L^2}\right) \le q \le 1$$

$$\langle s^2\rangle = \left\{\text{expression for } \langle s\rangle \text{ with } a \rightarrow 2a = 2|\ln(1/2)|^{1/2}\right\}$$

Figures 1 and 2 show the pdf and cdf for case (a) for a range of possible distributions in which the mean and variance of the v distribution was kept constant, and the u distribution parameters were varied. The units are arbitrary since only the dimensionless ratios appear in the distributions. The plots in display how increasing the mean (as well as the variance in this case) of the WR, u, affects the resulting distribution of s. Examination by the reader will confirm that the general features are intuitive, e.g. large WR results in low survivability, etc. The most probable values of s can be read off the peaks in Figure 1. Similarly, the quantiles may be read directly off the curves in Figure 2.

Figure 3 illustrates that one must take care when estimating performance based on point estimators of s (such as <SSPS>). In the figure, I have plotted various estimators of s for two distributions: one in which the variances of the u and v distributions are held constant and the means are varied and one in which the means are held constant and the variances are varied by widening the distance between upper and lower limits of the v distribution. The three plots labelled <SSPS>, SSPS(.75 quantile) and SSPS(.90 quantile) are the former case, the circles represent the latter. All are plotted against the median SSPS, which we saw above is simply the SSPS of the medians of u and v. The plot shows that the difference between the median and mean is particularly significant at low survivabilities. The circles plot the mean SSPS versus the median for a distribution with constant ratio of <u>/<v> but varying variance of u. The other two plots show the 75th and 90th percentiles for s, indicating how different the median is from levels with higher confidence.

The results above can be applied immediately, simply by computing the quantities of interest. We give two examples, in generic form, focusing on WR and CEP in turn.

WR often scales as the cube root of the yield[5], and one frequently needs to determine the relationhip between yield and SSPK at a specific confidence level for a particular system. Figure 4 shows SSPK as a function of normalized yield for a hypothetical system. Here the CEP had mean 1 and limits of ± 0.2 and the mean WR was varied from 1 to $\sqrt[3]{40} = 3.42$ with limits of $\pm 50\%$, thus representing a 40-fold range of yield. Four quantiles are shown, .5, .75, .90, .95, and .99. Suppose one wants the yield corresponding to SSPK = 0.9. If the .5 quantile is used, the "p50" plot indicates a yield of Y=6 is required to produce a median SSPK of .9. If one wants 90% confidence in 0.9 SSPK, one must use the "p90" curve, finding that a yield of about 28 is required, a factor of 4 larger. A 40-fold increase in yield is required for 0.95 confidence.

Generally a smaller CEP provides lower SSPS but requires more resources. Figure 5 shows how to pick a CEP that meets a specific level of confidence. The figure plots SSPK quantiles (0.5, 0.75, 0.90, 0.95, 0.99) versus CEP for a system characterized by <WR> = 1.0 $\pm 50\%$, and <CEP> varying as shown with constant limits of ± 0.20. For a median SSPK of 0.9 one sees that CEP is required to be about 0.55. Again, if 0.90 confidence of 0.90 SSPK is required, then a more stringent CEP of about 0.28 must be attained.

The point of the above examples is that one can be quantitative about confidence levels required for SSPK based on estimates of key system parameters.

The effect of uncertainty is shown in Figure 6, which plots the SSPK quantiles 0.5, 0.75, 0.90, 0.95 for constant <WR>=2, <CEP>=1, CEP range=± 0.20, but with the range of WR varying from 0.4 to 3.8. The result shows a constant median at $1-(1/2)^4 = 0.94$ but with widely varying quantiles.

Given an analytic form for the distribution of SSPK, one can calculate measures of how well the model reflects data from monte carlo or other first principle calculations. Several techniques are common, such as a chi-square test based on the observation that the difference between the actual and predicted SSPK values should be distributed as chi-square [e.g. 6] or examination of q-q plots [7]. We defer examples of these applications to specific treatments elsewhere.

3. Pdf and cdf for u uniform and v chi-squared.

Let u be distributed uniformly as in section 2 (eq. 3) but now let v be distributed as χ^2. Then if we define the variable ξ by $\xi = \frac{S^2(n-1)}{\sigma^2}$ with $S^2 \equiv \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1}$, then the pdf of ξ is given by (the χ^2-distribution)

$$f_\xi(\xi) = \frac{\xi^{\frac{n-1}{2}-1} e^{-\frac{\xi}{2}}}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} \quad (17)$$

The parameters σ and (n-1) are referred to as the standard deviation (actual) and the degrees of freedom respectively of the distribution and completely specify it. The quantity denoted by capital S, $S \equiv \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$, is the (maximum likelihood, unbiased) estimate of σ, in which each value of x_i represents the location of the i^{th} measurement of a weapon delivery system's impact point. The CEP is then 1.17741σ. Here we are interested in the distribution of $s = \left(\frac{1}{2}\right)^{g(u,\xi)}$ with $g(u,\xi) = \frac{u^2}{b\xi}$, $b \equiv \frac{(1.1774)^2\sigma^2}{n-1}$.

In this case, equation (5) for F becomes

$$F(s) = \iint_{(u,\xi):\, e^{g(u,\xi)} < s} f_u(u) f_\xi(\xi)\, du\, d\xi \tag{19}$$

Using $\xi = \frac{u^2 \ln(1/2)}{b \ln s}$, and equation (17) the cdf is

$$F(s) = \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})} \int_{u_L}^{u_H} \frac{du}{R_u} \int_{s'=0}^{s} \left(\frac{u^2 \ln(\frac{1}{2})}{b \ln s'}\right)^{\frac{n-1}{2}} \left(\frac{-1}{s' \ln s'}\right) e^{\frac{u^2 \ln(\frac{1}{2})}{2b \ln s'}}\, ds' \tag{20}$$

and the pdf is then a χ^2 probability integral

$$f(s) = \frac{1}{2^{\frac{1}{2}}\Gamma(\frac{n-1}{2})\, R_u} \left(\frac{-1}{s \ln s}\right) \left(\frac{b \ln s}{\ln(\frac{1}{2})}\right)^{\frac{1}{2}} \int_{\rho(u_L^2)}^{\rho(u_H^2)} \rho^{\frac{n}{2}-1} e^{-\rho}\, d\rho \tag{21}$$

where $\rho \equiv u^2 \frac{\ln(1/2)}{2b \ln s}$, yielding the result

$$f(s) = \frac{1}{\Gamma(\frac{n-1}{2})\, R_u} \left(\frac{1}{s|\ln s|^{\frac{1}{2}}}\right) \left(\frac{b}{2|\ln(\frac{1}{2})|}\right)^{\frac{1}{2}} \left[\Gamma\left(\frac{n}{2}, \frac{u_L^2 \ln(\frac{1}{2})}{2b \ln s}\right) - \Gamma\left(\frac{n}{2}, \frac{u_H^2 \ln(\frac{1}{2})}{2b \ln s}\right)\right] \tag{22}$$

where the incomplete gamma functions are defined in Box 2. We can now conveniently calculate F(s) and moments of the pdf by numerically integrating equation (22).

For illustration, we compute some distributions, examine the nature of the quantiles, and examine the tradeoff between yield and confidence in a particular SSPK for a hypothetical case.

Figure 7 shows pdfs for 6 illustrative distributions. The pdfs have 14 degrees of freedom and $\sigma = 1$, with various values for u_L and u_H chosen to illustrate low, high, and medium survivability cases. As in the uniform-uniform case, the modes of these distributions conform to intuition regarding low and high survivablility cases: for (WR/CEP) >> 1 the most probable SSPS goes to 0 and for (WR/CEP) << 1 the most probable SSPS approaches 1. For intermediate cases (WR/CEP) near 1, the pdfs can be bimodal as in the pdf labelled "pdf8".

Cumulative distribution functions are shown in Figure 8. These were obtained by straightforward numerical integration (on a MacII) of the pdfs in Figure 7. The interval length between evaluation points of the pdf was reduced until F(1) = 1.00 within about 1%, the number of points required ranging from 50 to several hundred. Confidence levels such as medians, 0.9 quantiles, etc., can be read directly off Figure 8 (or interpolated from the computed values) adequately for a 1% tolerance.

Figure 9 compares some of the quantiles to the median for the illustrative distributions. We also plot for comparison the average, <SSPS>, and the standard deviation of SSPS. There are large differences between the median (as well as the average) and high confidence levels such as the 0.9 quantile. For example, in the case in which the median SSPS is about 0.1 (the average SSPS is about 0.25) the 0.9 quantile is about 0.78. In terms of SSPK, the median SSPK is about 0.9 the average is about 0.75, and we have a 0.90 confidence that the SSPK is greater than only 0.22. The behavior is similar to the uniform-uniform case shown in Figure 3.

The relation between yield and SSPK is shown in Figure 10. CEP is taken to have a χ^2 distribution with $\sigma = 1$ and 14 degrees of freedom (<CEP> = 1.17741). The yield Y = 1 case is assumed to have WR distributed uniformly between 0.84 and 0.28 (in units of <CEP>). The effect of increasing yield was incorporated by increasing the mean WR by a factor $Y^{1/3}$ for Y = 2, 4, 5.6, 8, 10, 15, 20, 25, and 30 while maintaining a WR range equal to the mean WR (i.e. a 50% variation). We can use the plots to determine the yield required to produce a given SSPK with a given confidence level. For example, if we want to increase the 50% confident SSPK from 0.2 to 0.5 we must increase the yield by a factor of about 8. To increase the 90% confident SSPK from 0.05 to 0.5 requires an increase of yield of about a factor 24.

4. Treating v chi-square and u general.

In some cases, the distribution of u is obtained quasi-empirically. For example analysis of the effects of an earth penetrator weapon (EPW) involves propagation of strong shocks and earth motion through the ground, structural response of very hard targets, and the effects of various kinds of geology [8]. Numerical modeling of the deepndency of an EPW on parameters such as depth of burst and yield therefore typically result in quasi-empirical distributions from simulation of the effects of underlying distributions. We can incorporate such quasi-empirical results into the approach described here fairly easily.

We use equation (19)

$$F(s) = \iint\limits_{(u,\xi):\, e^{g(u,\xi)} < s} f_u(u)\, f_\xi(\xi)\, du\, d\xi \qquad (19)$$

but now equation (20) becomes simply

$$F(s) = \frac{1}{2^{\frac{n-1}{2}}\,\Gamma\!\left(\frac{n-1}{2}\right)} \int_{s'=0}^{s} ds' \left(\frac{-1}{s' \ln s'}\right) \left(\frac{\ln\!\left(\frac{1}{2}\right)}{b \ln s'}\right)^{\frac{n-1}{2}} \int_{u_{min}}^{u_{max}} du\, f_u(u)\, u^{n-1}\, e^{\frac{u^2 \ln\left(\frac{1}{2}\right)}{2 b \ln s'}} \qquad (23).$$

In this case we can perform the double integration numerically, using the quasi empirical values for $f_u(u)$. For moments such as the average, we can insert the appropriate power of s into the s integral in equation (23) before evaluation.

Acknowledgement

I would like to thank J. O'Dell for many helpful discussions on uncertainty analysis in weapon effectiveness studies.

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BOX 3

Case (b): $\frac{v_H}{v_L} \geq \frac{u_H}{u_L}$

$$f(s) = 0 \qquad s \leq s_1$$

$$f(s) = Q(s)\left[u_H^2 - v_L^2\left(\frac{\ln(s)}{\ln(1/2)}\right)\right] \qquad s_1 \leq s \leq s_3$$

$$f(s) = Q(s)\left[u_H^2 - u_L^2\right] \qquad s_3 \leq s \leq s_2$$

$$f(s) = Q(s)\left[v_H^2\left(\frac{\ln(s)}{\ln(1/2)}\right) - u_L^2\right] \qquad s_2 \leq s \leq s_4$$

$$f(s) = 0 \qquad s_4 \leq s$$

$$Q(s) \equiv -\frac{1}{4R_uR_v\ln(1/2)}\left(\frac{1}{s}\right)\left(\frac{\ln(1/2)}{\ln(s)}\right)^{3/2}$$

$$F(s) = 0 \qquad 0 \leq s \leq s_1$$

$$F(s) = \frac{1}{2R_uR_v}\left\{u_H^2\left(\frac{\ln(s)}{\ln(1/2)}\right)^{-1/2} - 2u_Hv_L + v_L^2\left(\frac{\ln(s)}{\ln(1/2)}\right)^{1/2}\right\} \qquad s_1 \leq s \leq s_3$$

$$F(s) = \left[\left(\frac{\ln(s)}{\ln(1/2)}\right)^{-1/2} - \frac{u_L}{v_H}\right]\left[\frac{u_H^2 - u_L^2}{2R_uR_v}\right] + F(s_3) \qquad s_3 \leq s \leq s_2$$

$$F(s) = \frac{1}{2R_uR_v}\left\{u_Hv_H + \frac{u_L^2v_H}{u_H} - v_H^2\left(\frac{\ln(s)}{\ln(1/2)}\right)^{1/2} - u_L^2\left(\frac{\ln(s)}{\ln(1/2)}\right)^{-1/2}\right\} + F(s_2) \qquad s_2 \leq s \leq s_4$$

$$F(s) = 1 \qquad s_4 \leq s \leq 1$$

Figure 1

PROBABILITY DENSITY FUNCTION FOR SSPS FOR U & V UNIFORMLY DISTRIBUTED

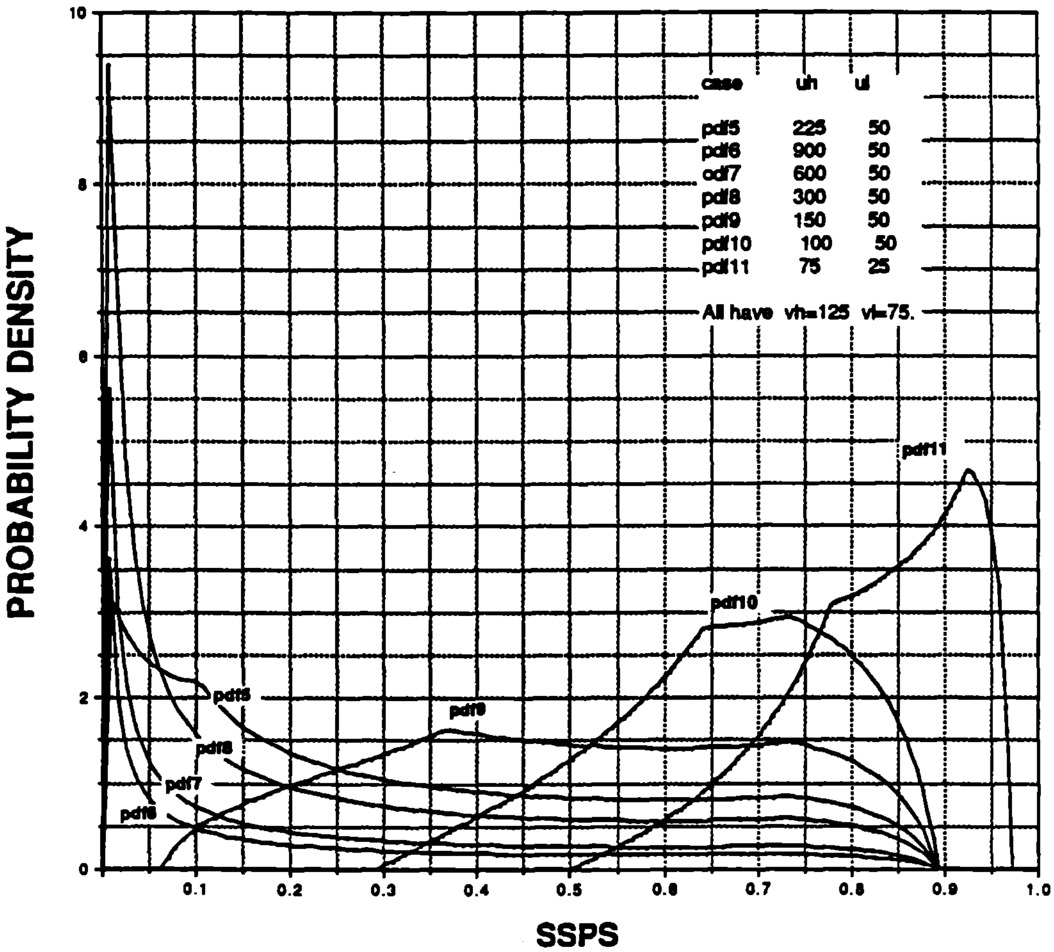


Figure 2
CUMULATIVE DISTRIBUTION FUNCTIONS FOR U AND V UNIFORMLY DISTRIBUTED

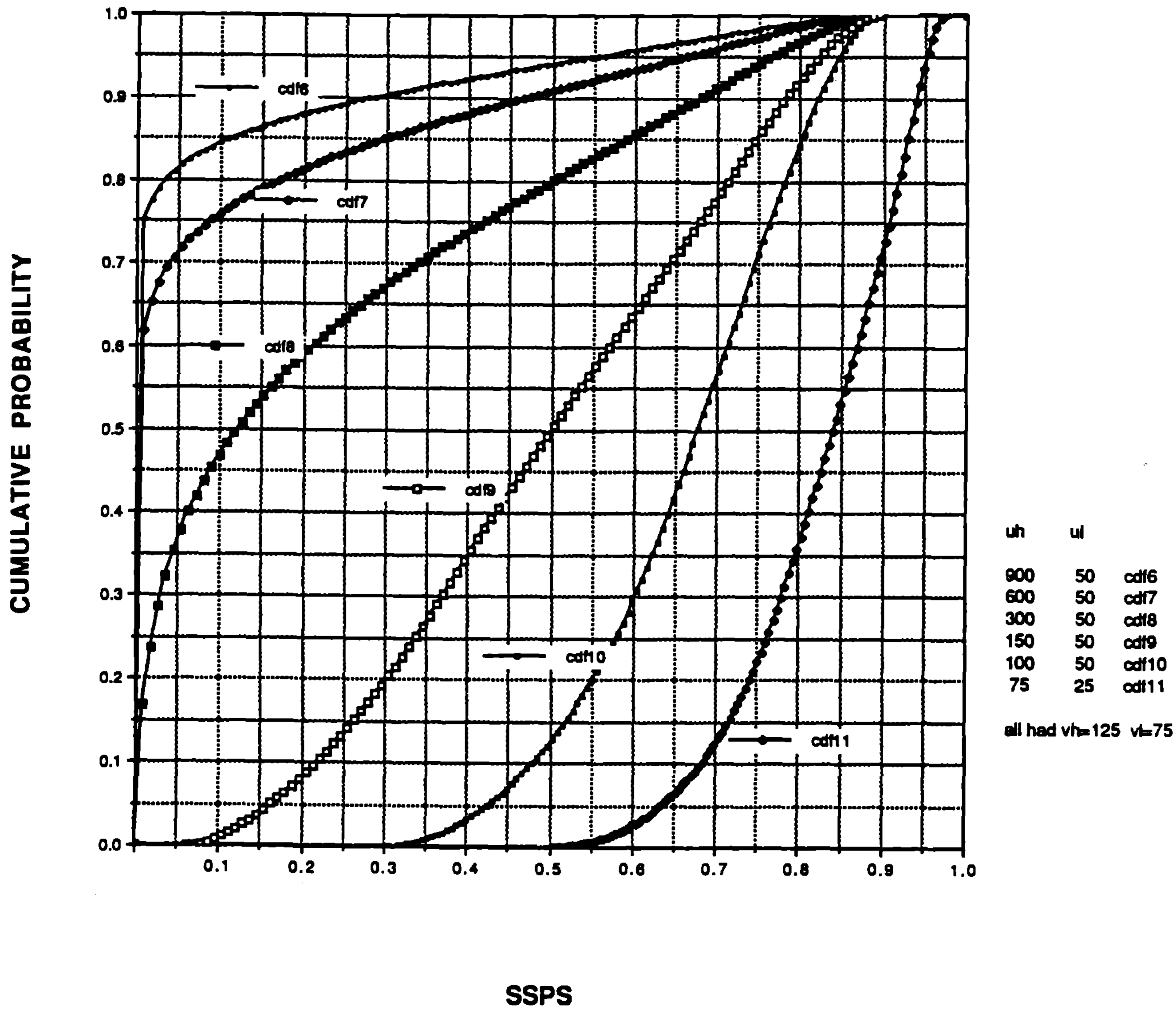


Figure 3

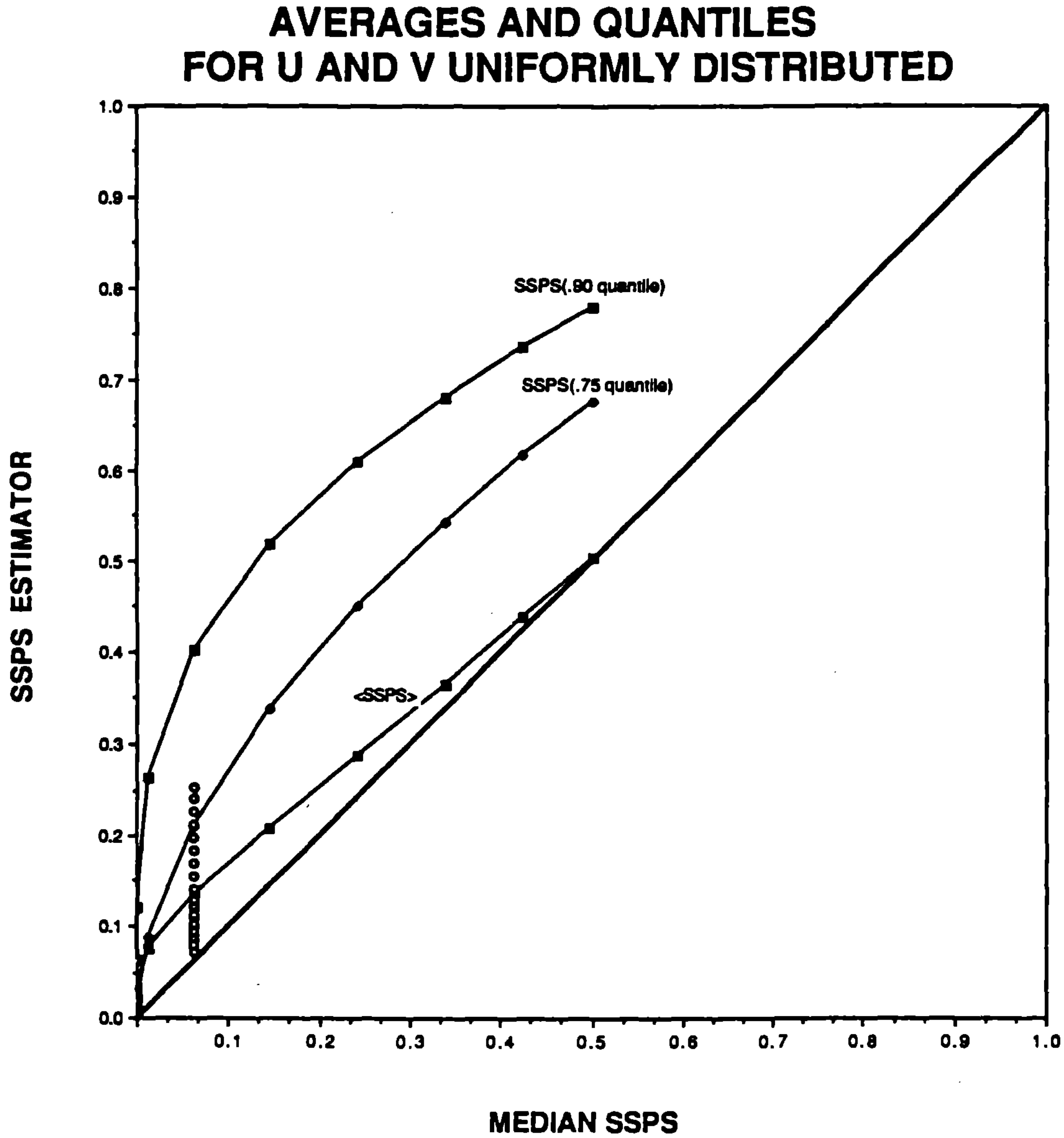

AVERAGES AND QUANTILES
FOR U AND V UNIFORMLY DISTRIBUTED
SSPS(.90 quantile)
SSPS(.75 quantile)
<SSPS>
SSPS ESTIMATOR
MEDIAN SSPS
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Figure 4

EXAMPLE: YIELD AND SSPK FOR U AND V UNIFORMLY DISTRIBUTED

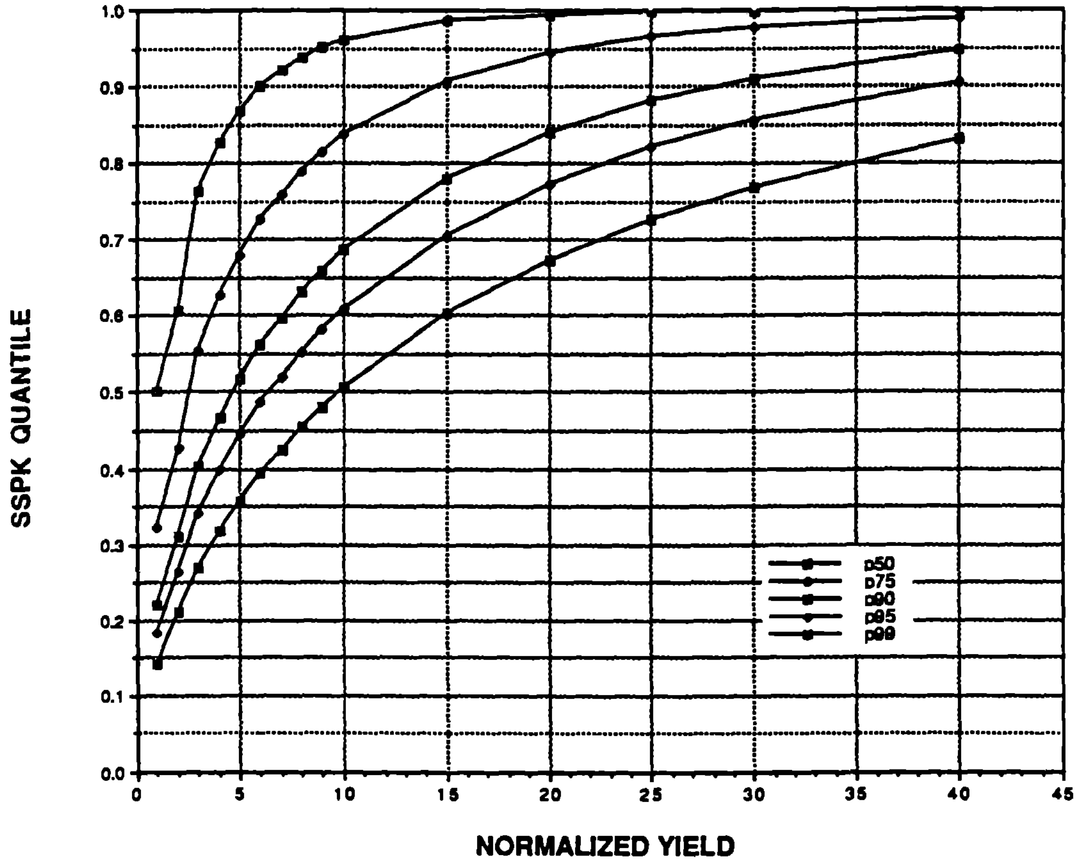


Figure 5

EXAMPLE: SSPK AND CEP
FOR U AND V UNIFORMLY DISTRIBUTED

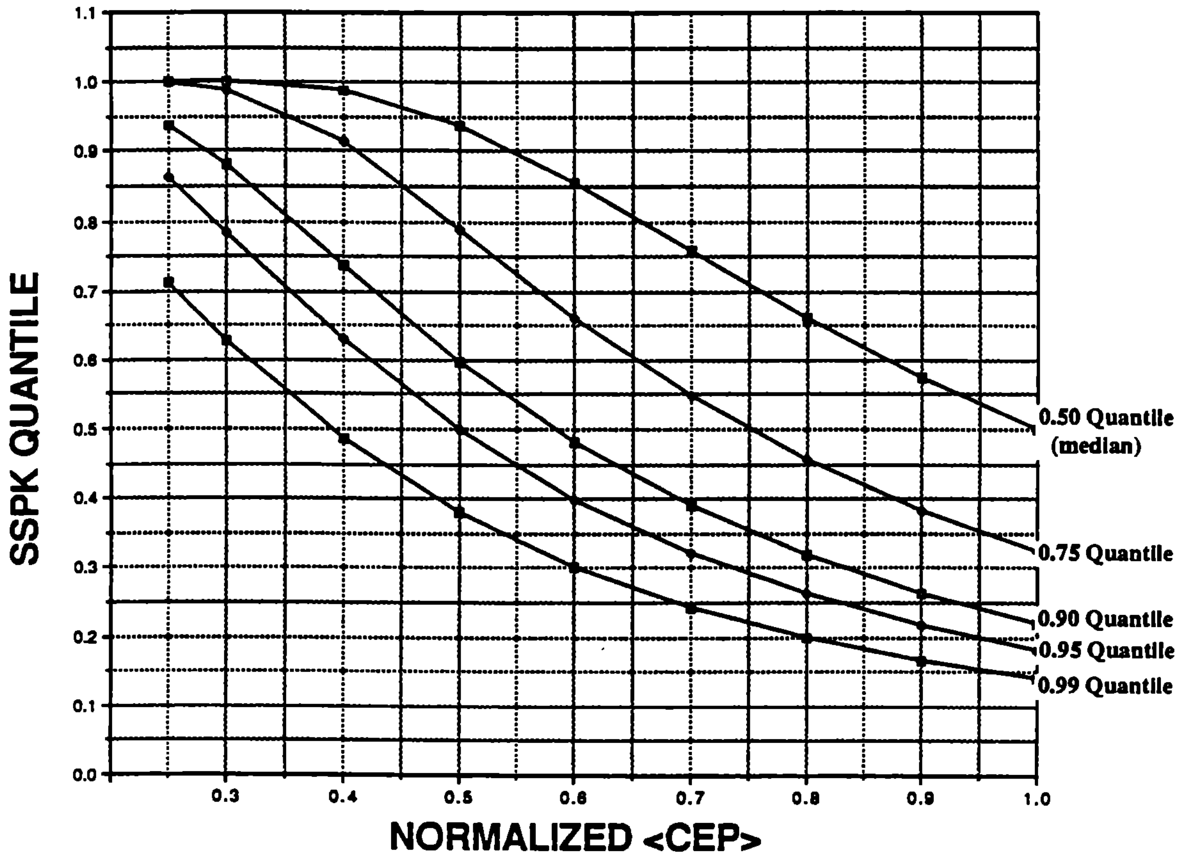
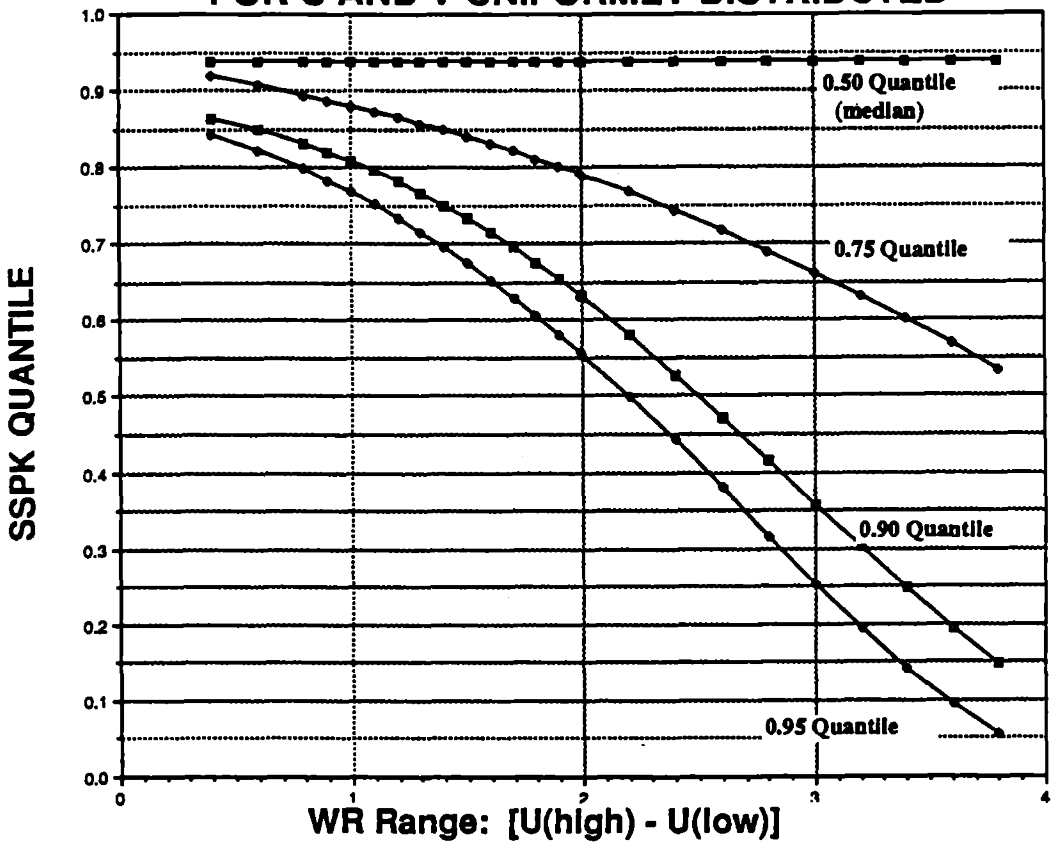


Figure 6

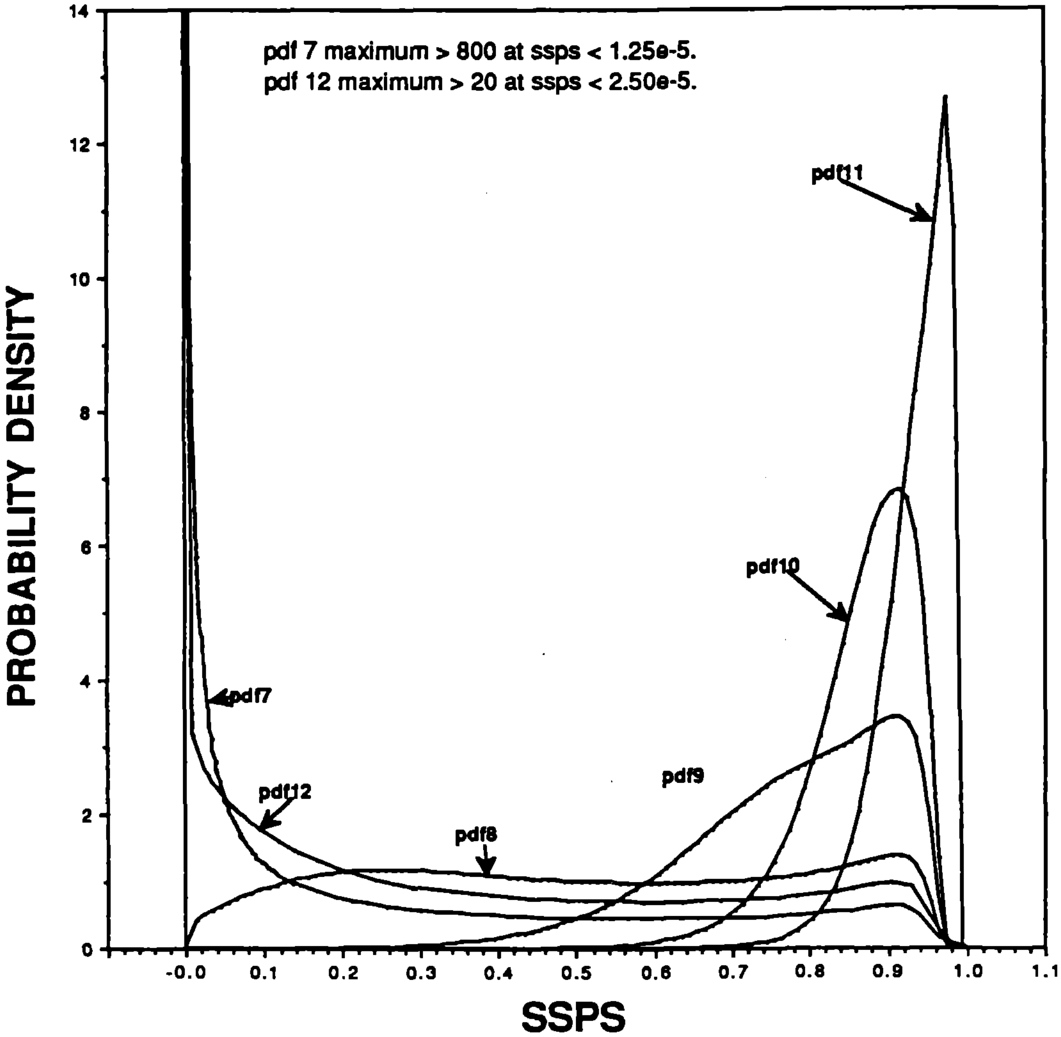
EXAMPLE: WR AND SSPK
FOR U AND V UNIFORMLY DISTRIBUTED



(In units of <cep> = 1)

Figure 7

PDFS FOR SSPS
FOR U UNIFORM AND V CHI-SQUARE

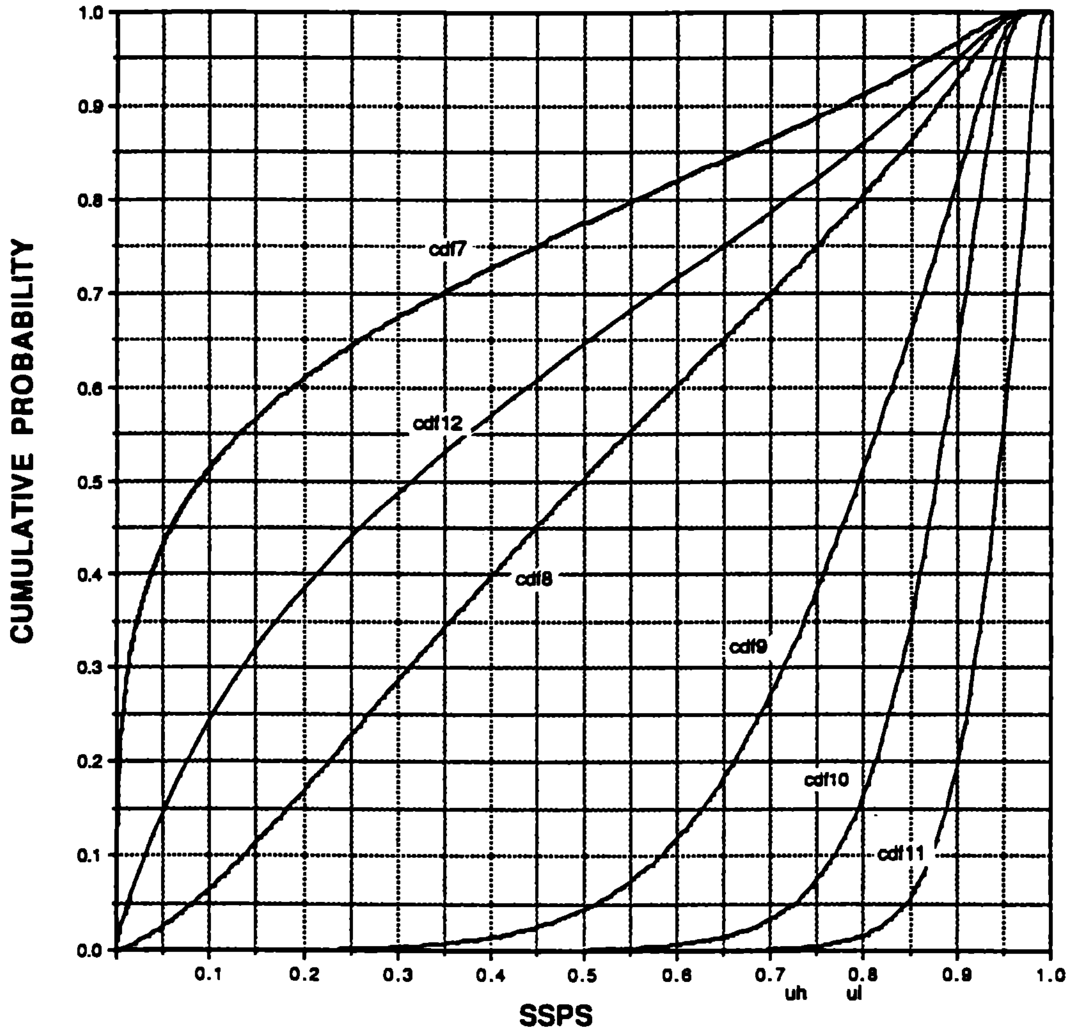


uh	ul	
6.00	.50	pdf7
4.00	.50	pdf12
3.00	.50	pdf8
1.50	.50	pdf9
1.00	.50	pdf10
.75	.25	pdf11

All have <cep> = 1.1774, (i.e. sigma = 1.00), and n - 1 = 14 degrees of freedom.

Figure 8

SSPS CDFS
FOR U UNIFORM AND V CHI-SQUARE



uh	ul	
6.00	.50	cdf7
4.00	.50	cdf12
3.00	.50	cdf8
1.50	.50	cdf9
1.00	.50	cdf10
.75	.25	cdf11

Figure 9

AVERAGES AND QUANTILES FOR U UNIFORM AND V CHI-SQUARE

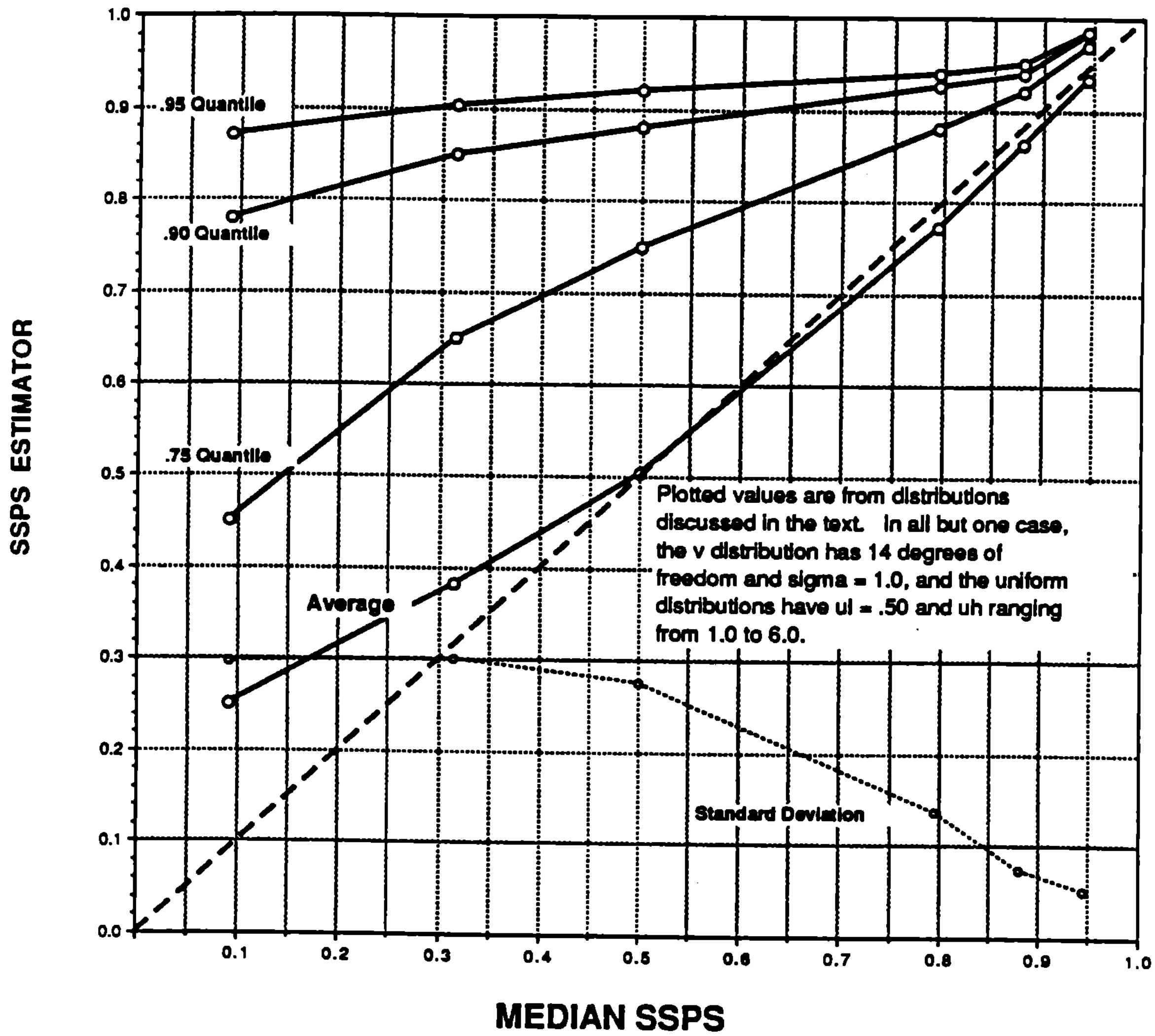


Figure 10

EXAMPLE: YIELD AND SSPK FOR U UNIFORM & V CHI-SQUARE